A Gentle Introduction to Surgery Theory: The Mathematics of Modifying Manifolds

Gabriel Solis

November 2024

Abstract

In this paper, we give a very gentle introduction of surgery theory, taking primarily a geometric approach to provide intuition. Section 1 motivates surgery theory. In Section 2, we then provide a stable foundation of fundamental concepts on manifolds and diffeomorphisms that will guide our later discussions. We then discuss in Section 3 Morse functions and handle decompositions that provide the tools of surgery. This finally leads us to actually formalize surgery and its connection to handle attachment. The paper finishes with a note on different applications of surgery theory.

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1 Introduction

In this paper, we set out to explore the foundations of surgery theory, its applications, and where it might be going in the future. Surgery theory, as the name suggests, is all about cutting up manifolds—but don't worry, no scalpel experience is required. It's the mathematical version of remodeling a house: carefully removing one part and replacing it with something better, all while keeping the structure intact. It's a surprisingly versatile tool, with applications ranging from spacetime physics to knot theory and even to Grigori Perelman's million-dollar solution to the Poincaré conjecture.

Now that we've motivated the importance of surgery theory, let us begin our exploration. This exposition adopts a geometric perspective, emphasizing diagrams and visual examples to build our intuition. We will start by laying the groundwork with diffeomorphisms and smooth manifolds. From there, we will introduce Morse theory and handle decompositions, essential tools for understanding the surgical process. Finally, we will take a closer look at the surgical process itself and its wide-ranging applications.

2 Fundamental Concepts

In this section, we develop the central concepts needed for surgery theory. We begin by introducing manifolds and the notion of smoothness. We then explore diffeomorphisms - maps between manifolds that preserve both their smooth structure and topological properties. Finally, we synthesize these ideas to discuss smooth manifolds.

2.1 Manifolds

Loosely, an n-dimensional topological manifold M is a Hausdorff space that is locally Euclidean, meaning each point in M has a neighborhood topologically equivalent to n-dimensional Euclidean space. The formal definition follows.

Definition 1. An n-dimensional *manifold* M is a second countable, Hausdorff topological space such that for all points $p \in M$, there exists an open neighborhood N_p of p such that N_p is homeomorphic to the Euclidean open n-ball, $B^n := \{x \in \mathbb{R}^n \mid \|x\| < 1\}$.

To describe manifolds rigorously, we need the concepts of coordinate systems, atlases, and transition maps that will come in handy when we talk about smooth manifolds.

Definition 2. Let M be a topological space and $U \subset M$ an open set. Let $V \subset \mathbb{R}^n$ be open. A homeomorphism $\phi : U \to V$, $\phi(u) = (x_1(u), \ldots, x_n(u))$, is called a *coordinate system* on U, and the functions $x_1, \ldots, x_n$ the *coordinate functions*. The pair (U, ϕ) is called a *chart* on M. The inverse map ϕ^{-1} is a *parameterization* of U.

We visualize the idea of a *chart* in Figure 1 for $M = S^2$. We can see that the *chart* corresponds to a flat sheet that is wrapped around S^2.

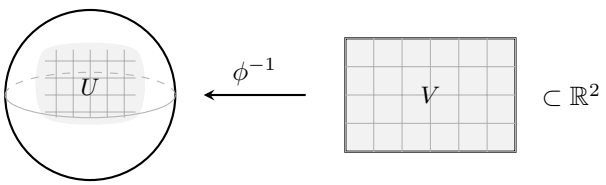


Figure 1: A local coordinate chart ϕ for S^2, mapping a patch on the sphere (left) to an open subset of $\mathbb{R}^2$ (right) via the inverse map ϕ^{-1}.

Definition 3. An *atlas* on M is a collection of charts $\{U_\alpha, \phi_\alpha\}$ such that $\{U_\alpha\}$ cover M.

By definition, every point on a manifold lies in a chart, and thus every manifold has an atlas. However, an atlas alone is insufficient for applying calculus to manifolds. This requires a stronger condition, which we will address through smooth manifolds after diffeomorphisms.

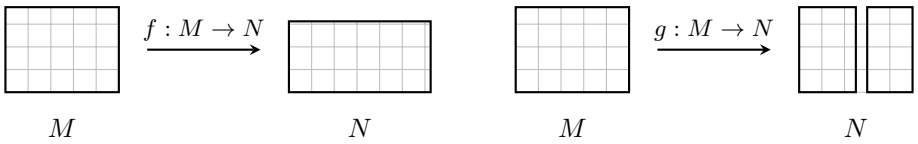
2.2 Diffeomorphisms

When studying manifolds, we must go beyond topology. Homeomorphisms preserve topological equivalence but may not maintain differentiability across charts. To work locally in Euclidean space and perform cutting and gluing in surgery theory, we need maps that preserve derivatives. This leads us to diffeomorphisms, a stronger equivalence. Let's start by defining smooth functions.

Definition 4 (Smooth Function)**.** Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets, and let $F : U \to V$ be a function such that $F(y) = (F^1(y), F^2(y), \ldots, F^m(y))$, where each $F^k : U \to \mathbb{R}$ represents the k-th component of F. The function F is *smooth* if each component function F^k has continuous partial derivatives of all orders.

With smooth functions introduced, we can now define our notion of "similar enough" that is stronger than homeomorphism

Definition 5 (Diffeomorphism)**.** A function $f : M \to N$ is a *diffeomorphism* if it is a bijection, is smooth, and has a smooth inverse f^{-1}.



(a) Diffeomorphism (b) Non-Diffeomorphism

Figure 2: (a) A diffeomorphism $f : M \to N$ that smoothly stretches and compresses M, preserving differentiable structure. (b) A non-diffeomorphism $g : M \to N$ introducing a discontinuity in N, violating smoothness.

Figure 2 illustrates examples of both a diffeomorphism and a non-diffeomorphism. Observe that smoothness implies differentiability, which in turn implies continuity. As these properties hold for both f and f^{-1}, every diffeomorphism is necessarily a homeomorphism. Thus, a diffeomorphism preserves not only the topological structure but also the differentiable structure between M and N.

Lemma 1. *Two spaces M, N being diffeomorphic ($M \sim_{diff} N$) is an equivalence relation.*

It is straightforward to verify that diffeomorphism satisfies reflexivity, symmetry, and transitivity. We leave the details to the reader to conserve space for more central topics.

2.3 Smooth Manifolds

Definition 6. Let $(\mathcal{U}_\alpha, \phi_\alpha)$ and $(\mathcal{U}_\beta, \phi_\beta)$ be charts on a manifold such that $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$. The *transition map* from ϕ_α to ϕ_β is the map $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to \phi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$

Observe that a transition map is the composition of homeomorphisms and is therefore a homeomorphism. Now we'll see that the smoothness of these transition maps is a critical requirement for defining smooth manifolds. Figure 3 illustrates a comparison between a smooth manifold and a non-smooth manifold.

Definition 7. A second countable, Hausdorff topological space M is a *smooth manifold* if it admits an atlas $(\mathcal{U}_\alpha, \phi_\alpha)\}, \phi_\alpha : \mathcal{U}_\alpha \to \mathbb{R}^n, n \in \mathbb{N}$, where all transition maps are smooth (C^∞) diffeomorphisms.



Figure 3: (a) The 2-dimensional sphere S^2. (b) The cone is non-smooth at the apex

3 Tools of Surgery

In the previous section, we saw that smooth manifolds inherit the topological and differentiable properties of Euclidean space. Now, we explore smooth functions mapping manifolds to the real line. Like calculus uses critical points to study curves and surfaces, Morse functions—smooth functions on manifolds—reveal their topology via critical points. The next section visualizes this intuition on the torus T^2.

3.1 Morse Functions

We start with the two-dimensional torus $M = T^2$ and consider the height function $f : M \to \mathbb{R}$, which assigns to each point $x \in M$ its height above a horizontal plane, as illustrated in Figure 4. The figure provides a visual representation of the height values, with the height axis shown on the right.

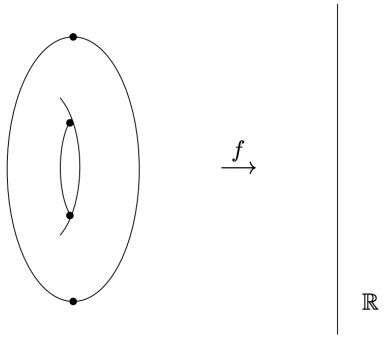


Figure 4: A Torus with Critical Points and a Height Function f.

Each real number a has a preimage, $f^{-1}(a)$, which we refer to as a *level set*, consisting of all points $x \in M$ with height a. Accordingly, the *sublevel set* consist of all points with height at most a,

$$M^a = f^{-1}((-\infty, a]) = \{x \in M \mid f(x) \leq a\}$$

We can also use the height interpretation to "unveil" our manifold, starting from nothing and slowly accumulating slices as we go up in height (refer to Figure 5).

Comparing M^a and M^b for $a, b \in \mathbb{R}$ with $a < b$, we ask when they share the same topology and when they differ. The answer lies in critical points $p \in M$ where $\nabla f = 0$, which mark topological changes in sublevel sets M^a and M^b when $a < f(p) < b$. Each critical point has an index measuring the directions f decreases. For instance, at the first critical point (a minimum, index 0), the sublevel set becomes a disk. Subsequent indices reflect how the torus curves at each point, leading to *handle decompositions*, where each critical point adds a specific handle. This is explored in the next section.

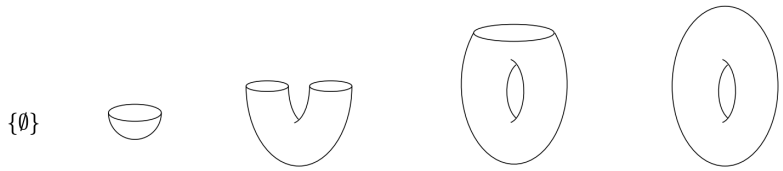


Figure 5: Unveiling a Torus

3.2 Handle Decompositions

Definition 8. The process of attaching a j-handle to a smooth n-manifold M transforms M into $M \cup_f H^j$, where $0 \leq j \leq n$, $f : S^{j-1} \times D^{n-j} \to \partial M$ is a diffeomorphism, and $H^j = D^j \times D^{n-j}$ is a j-handle.

To understand this definition, we will look back at our unveiling procedure. However, before we proceed, we'll introduce the following lemma that will guide our discussion on handle decomposition.

Lemma 2. *Let M be a compact boundaryless manifold. Let $f : M \to \mathbb{R}$ be a Morse function. Assuming that the critical points $\{p_i\} \subset M$ are such that $f(p_1) < f(p_2) < \cdots < f(p_k)$, and provided $t_0 < f(p_1) < t_1 < f(p_2) < t_2 < \cdots < f(p_k) < t_k$, $f^{-1}[t_{j-1}, t_j]$ is diffeomorphic to $(f^{-1}(t_{j-1}) \times [0,1]) \cup_g H^{i(p_j)}$, where $i(x)$ gives the index of x. Here g is a diffeomorphism which determines where the handle $H^{i(p_j)}$ is attached on ∂M.*

We will not prove this lemma but instead demonstrate its role in our torus unveiling. We begin with the empty set and pass the first critical point, indexed at 0. At this stage, we attach $H^0 = D^0 \times D^2$ to the empty set along $g(S^{-1} \times D^2) = \emptyset$, resulting in $H^2 = g(D^0 \times D^2) = g(D^2)$, an embedded disk (see Figure 6).

Next, passing the second critical point, indexed at 1, we attach $H^1 = D^1 \times D^1$ along $h(S^0 \times D^1) \in \partial M^a$. This step adds a strip between two sections of the disk's boundary, forming a figure homeomorphic to the bottom half of the torus. At the third critical point, also indexed at 1, we attach another $H^1 = D^1 \times D^1$ along $m(S^0 \times D^1)$, connecting the disconnected boundary circles. The resulting figure is homeomorphic to a torus missing its top. Finally, at the fourth critical point, indexed at 2, we attach $H^2 = D^2 \times D^0$ along $n(S^1 \times D^0)$, capping the boundary circle and completing the torus.

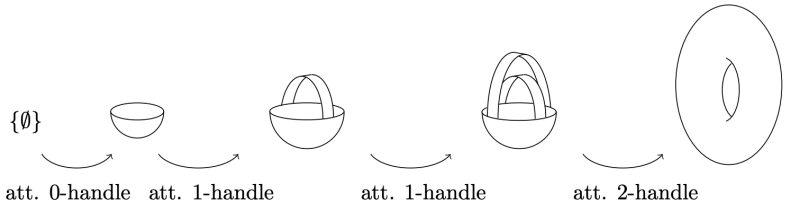


Figure 6: Topological Progress of M^a via Handle Attachment

4 Surgery Theory

We now have the mathematical requisites to briefly describe surgery theory. Informally, surgery theory is the study of the effects of "slicing" a manifold through an operation called *surgery*. More precisely, surgery is the transformation of one manifold into another by cutting out $S^k \times D^{n-k}$ and replace it with $D^{k+1} \times S^{n-k-1}$ for some k.

4.1 Mathematical Formalism of Surgery

The formal definition of surgery is:

Definition 9 (Surgery on a Smooth Manifold)**.** Given a smooth n-manifold M, an integer $0 \leq k \leq n$, and a diffeomorphism $f : S^k \times D^{n-k} \to M$, the manifold M' produced from surgery is

$$M' = \Big(M - f\big(S^k \times D^{n-k} - S^k \times S^{n-k-1}\big)\Big) \cup_g \big(D^{k+1} \times S^{n-k-1}\big), \tag{1}$$

where the gluing map g is defined as $g = f|_{S^k \times S^{n-k-1}}$.

Intuitively, we can decompose it into two key operations. First, we excise a portion of our original manifold M, specifically the image of $S^k \times D^{n-k}$ under f, while being careful to preserve its boundary. This excision is captured by the term $M - f(S^k \times D^{n-k} - S^k \times S^{n-k-1})$. Second, we perform a strategic replacement, attaching $D^{k+1} \times S^{n-k-1}$ along the preserved boundary using the gluing map g.

Example 1. Let's consider the simplest non-trivial example: surgery on the circle S^1 (see Figure 7). Surgery on S^1 begins by excising $S^0 \times D^1$ (highlighted in red), which consists of two disjoint intervals, leaving two disconnected black arcs. These intervals are replaced with $D^1 \times S^0$ (also red) and attached to the arcs via a gluing map $\cup_g$. The choice of the gluing map determines the outcome: in case (a), the intervals are joined to preserve the original orientation, reconstructing S^1; in case (b), the map connects endpoints differently, creating two disjoint S^1's. The figure demonstrates how the surgery outcome depends on the gluing map.

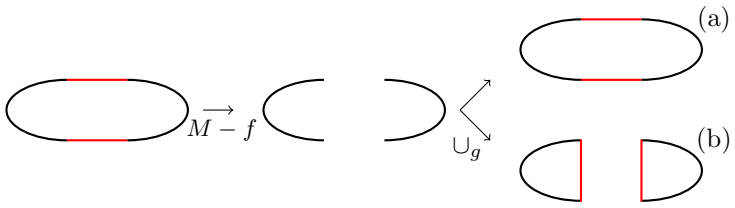


Figure 7: Surgery on S^1 yields either (a) an unchanged S^1 or (b) two disjoint S^1.

We'll end this section by formalizing the connection between surgery and handle attachments by considering the following lemma:

Lemma 3. *Let M be an n-manifold. Let $0 \leq k \leq n$ and let $f : S^{k-1} \times D^{n-k} \to \partial M$ be a diffeomorphism. Let handle(M, k) be the manifold produced from M via k-handle attachment with the above parameters. Let surgery$(\partial M, k-1)$ be the manifold produced from ∂M via $(k-1)$-surgery using the same diffeomorphism.*

$$\partial \mathit{handle}(M, k) = \mathit{surgery}(\partial M, k-1).$$

This result shows that a k-handle attachment on an n-manifold M is equivalent to a $(k-1)$-surgery on its boundary, ∂M. For example, in Section 3.2, we saw how the torus is constructed via sequential handle attachments. This lemma generalizes such constructions, bridging surgery theory and handle decompositions across arbitrary dimensions.

4.2 Application: The h-Cobordism Theorem

The h-Cobordism Theorem is a notable application of surgery theory. While the proof is too long to share here, we can describe the theorem and a fundamental result of it. We begin by defining a cobordism:

Definition 10. An n-dimensional *cobordism* $(W; M_0, f_0; M_1, f_1)$ consists of a compact n-dimensional manifold W, closed $(n-1)$-dimensional manifolds M_0, M_1, a decomposition of the boundary of W as $\partial W = V_0 \sqcup V_1$, and diffeomorphisms $f_i : M_i \to V_i$ for $i = 0, 1$.

An example of a cobordism would be a *pair of pants*, a cobordism between S^1 and $S^1 \sqcup S^1$.

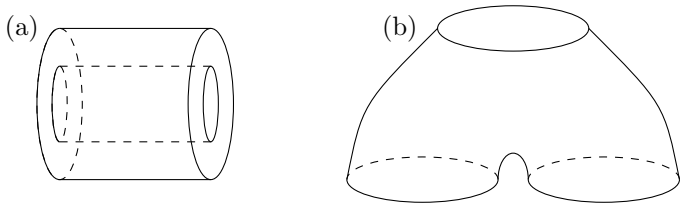


Figure 8: (a) A trivial cobordism. (b) A pair of pants.

The h-Cobordism Theorem is interested in when cobordisms are equivalent:

Definition 11. Two cobordisms $(W; M_0, f_0; M_1, f_1), (W'; M_0', f_0'; M_1', f_1')$ are *diffeomorphic relative to* M_0 if there exists a diffeomorphism $F : W \to W'$ such that $F \circ f_0 = f_0'$.

Cobordisms are often denoted by $(W; M_0, M_1)$ or $(W; V_0, V_1)$, where V_0, V_1 are the decomposed boundary components. Of paramount importance is the notion of an h-cobordism:

Definition 12. A cobordism $(W; V_0, V_1)$ is an *h-cobordism* if the inclusions $V_0 \to W$ and $V_1 \to W$ are homotopy equivalences; that is, their composition is homotopic to identity.

Theorem 4. *Let $(W; V_0, V_1)$ be any h-cobordism with $\dim(W) \geq 5$. If M and N are closed and simply connected, then $(W; V_0, V_1)$ is diffeomorphic to $(V_0 \times [0,1]; V_0 \times \{0\}, V_0 \times \{1\})$.*

The cobordism $(M \times [0,1]; M \times \{0\}, M \times \{1\})$, where M is a closed manifold, is also called the *trivial cobordism* (see figure (8), with M an annulus).

The h-Cobordism Theorem is powerful in classifying high-dimensional simply connected manifolds. For instance, the Poincaré Conjecture for $n \geq 5$ is a corollary of the theorem:

Theorem 5. *All simply connected closed n-manifolds are homeomorphic to S^n for $n \geq 5$.*

The Poincaré Conjecture actually holds for $n \geq 3$, but lower-dimensional cases are notoriously difficult and were proven many decades after the h-Cobordism Theorem. The case of $n = 3$ was proven by Grigori Perelman in 2003 using Ricci flow with surgery.

5 Conclusion

As we conclude our journey through surgery theory, we hope to have inspired a sense of admiration for the powerful mathematical tools it offers for modifying and classifying manifolds. These tools have broad practical and theoretical applications, from spacetime topology in physics[1] to the classification of high-dimensional manifolds.

So where is surgery theory heading? From the exploration of 4-dimensional manifolds to topics like Ricci curvature in differential geometry, surgery theory has been and continues to be a powerful tool for understanding the mathematical universe! We're excited for what's next in the study of surgery theory.

[1]Wang, J., Wen, X.-G., & Yau, S.-T. (2019). *Quantum statistics and spacetime topology: Quantum surgery formulas. arXiv preprint arXiv:1901.11537.* Retrieved from `https://arxiv.org/abs/1901.11537`.

References

[1] Andrew R. Mackie-Mason. *The h-Cobordism Theorem*. 2011. URL: https://www.math.uchicago.edu/~may/VIGRE/VIGRE2011/REUPapers/MacKieMason.pdf.

[2] Remy Bohm. *Morse Theory and Handle Decompositions*. 2019. URL: https://math.uchicago.edu/~may/REU2019/REUPapers/Bohm.pdf.

[3] Jonathan Marty. *Surgery Theory*. 2021. URL: https://www.uvm.edu/~smillere/TProjects/JMarty21f.pdf.

[4] Josep Pérez Díez. "On the h-cobordism theorem and its applications". MA thesis. Universitat de Barcelona, 2022. URL: https://diposit.ub.edu/dspace/bitstream/2445/198300/1/tfm_perez_diez_josep.pdf.

[5] Jonathan Rosenberg. *Surgery theory today - what it is and where it's going*. URL: https://math.umd.edu/~jmr/surgery.pdf.